A mixed Eulerian-Lagrangian scalar transport model

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Is it possible to construct an exact scalar transport model for an idealized turbulence analog?

Assumption:
Turbulent transport is dominated by the vortical structures making up the flows
A Two-Dimensional Model of 3D Turbulence:

Point (line) vortex:

\[ u_\theta \sim \frac{1}{r} \]

\[ u_r = u_z = 0 \]

\[ \mathbf{u}(\mathbf{x}) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi|\mathbf{x} - \mathbf{x}_k|} \left( \hat{z} \times \left( \mathbf{x} - \mathbf{x}_k \right) \right) \]

Gruchalla et al. (2009)

Mininni et al. (2008+)

Rast & Pinton (2009)
Point vortex simulations:

\[
\mathbf{u}(\mathbf{x}) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi|\mathbf{x} - \mathbf{x}_k|} \left( \hat{z} \times \left( \mathbf{x} - \mathbf{x}_k \right) \right)
\]

\[u_\theta \sim 1/r\]
\[u_r \sim 0\]

Merger of close vortices
Stirring by vortex creation

“Trapping” events at all scales dominate Lagrangian single point and pair dispersion statistics
Pair Dispersion (trapping dominates scaling):

$$r^2 = r_0^2 + (t - t_d)^\alpha$$

Delay time uniform distribution then

$$\langle r^2 \rangle \propto \int (t - t_d)^\alpha dt \propto t^{\alpha+1}$$

With delay uniform distribution of delay times:

Underlying Batchelor scaling: \( r^2 \sim t^2 \) \( \Rightarrow \) \( \langle r^2 \rangle \propto t^3 \)

Underlying Richardson scaling: \( r^2 \sim t^3 \) \( \Rightarrow \) \( \langle r^2 \rangle \propto t^4 \)Observed
Constructing a transport model:

Consider transport of a scalar quantity, \( c \).

\[
\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = S(x,t) \quad \quad c(x,t) = \int S(x',t') \ G(x,t \mid x',t') \ dx' \ dt'
\]

\[
\langle c(x,t) \rangle = \int S(x',t') \ P(x,t \mid x',t') \ dx' \ dt'
\]

Measure \( P(x,t \mid x',t') \):

If one had this completely, the problem would be solved.

Instead take: \( P(x,t \mid x',t') = \frac{P(r,t)}{2\pi r} \) (isotropy)

From Lagrangian trajectories: \( P(r,t) \), the probability of traveling an Eulerian distance \( r \) in time \( t \) along a Lagrangian path.
\[ P(r,t) \sim re^{-r^2/\sigma(t)^2} \]

\[ \sigma^2 \sim \frac{t}{2} \quad \text{for } t > t_L \]

\[ \sigma^2 \sim t^2 \quad \text{for } t < t_L \]
What about the variance of $c$?

$$\langle c^2(x,t) \rangle = \int S(x_1,x_2,t_1,t_2) P(x,x,t,t | x_1,x_2,t_1,t_2) dx_1 dx_2 dt_1 dt_2$$

$$P(x,x,t,t | x_1,x_2,t_1,t_2) = P(x_1,x_2,t_1,t_2 | x,x,t,t)$$

Reversible, statistically steady flow

$$P(x,x,t,t | x_1,x_2,t_1,t_2) = \frac{1}{4\pi r \Delta r} P(r,t) P(\Delta r, \Delta t | r,t)$$

Pair dispersion problem generalized to include time.
Toward a turbulent transport model:

1. Model turbulent transport using the statistics of Lagrangian trajectories in a point vortex flow.

2. Identify coherent vortical structures in simulations of real three-dimensional turbulence.

3. Use the Lagrangian statistics of in presence of these coherent structures in place of those due to point vortices in a transport model.

For LES must also:

Relate the statistics of the coherent vortical structures to the large scale flow producing them.