Asymptotic Approaches for Rotationally Constrained Convective Flows

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Navier-Stokes Equations: GAFD

- Generic non-dimensionalization: \( L, U, \Delta T, P \)

\[
D_t \mathbf{u} + Ro^{-1} \hat{\mathbf{z}} \times \mathbf{u} = -Eu \nabla p + \Gamma T \hat{\mathbf{r}} + Re^{-1} \nabla^2 \mathbf{u} + \mathbf{S}
\]

\[
D_t T = Pe^{-1} \nabla^2 T
\]

\[
\nabla \cdot \mathbf{u} = 0
\]

where \( D_t := \partial_t + \mathbf{u} \cdot \nabla \) with \((u, p, T)\) for velocity, pressure & temperature fields.

- Non-dimensional Parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rossby Number</td>
<td>( Ro = \frac{U}{2\Omega L} )</td>
</tr>
<tr>
<td>Euler Number</td>
<td>( Eu = \frac{P}{\rho_0 U^2} )</td>
</tr>
<tr>
<td>Buoyancy Number</td>
<td>( \Gamma = \frac{g \alpha \Delta T L}{U^2} )</td>
</tr>
<tr>
<td>Ekman Number</td>
<td>( Ek = \frac{Ro}{Re} = \frac{\nu}{2\Omega L^2} )</td>
</tr>
<tr>
<td>Reynolds Number</td>
<td>( Re = \frac{UL}{\nu} )</td>
</tr>
<tr>
<td>Péclet Number</td>
<td>( Pe = \frac{UL}{\kappa} )</td>
</tr>
</tbody>
</table>
Navier-Stokes Equations: Rotationally Constrained Flows, $Ro \ll 1$

- For $Ro \ll 1$ turbulence challenge compounded

$$D_t u + Ro^{-1} \mathbf{\hat{z}} \times u = -Eu \nabla p + \Gamma T \mathbf{\hat{r}} + Re^{-1} \nabla^2 u + S$$
$$D_t T = Pe^{-1} \nabla^2 T$$
$$\nabla \cdot u = 0$$

- NSE stiff PDE, \exists fast inertial waves & slow geostrophically balanced eddies

**Fast Inertial Waves**

$$\omega_{fast} \sim Ro^{-1} \frac{k_z}{\sqrt{k_{\perp}^2 + k_z^2}}$$

of secondary importance

**Geostrophic Eddies/Slow Waves**

$$\omega_{slow} \sim \mathcal{O}(1)$$

$$Ro^{-1} \mathbf{\hat{z}} \times u \approx -Eu \nabla p, \quad \nabla \cdot u = 0 \Rightarrow \mathbf{\hat{z}} \cdot \nabla (u, p) \approx 0$$

Proudman-Taylor Thm (1916,1923) motions are inherently columnar
Planetary Scale Rotationally Constrained Convection $Ro \ll 1$

turbulence primary driver for geomagnetic field

large-scale flow generation

$Ro \sim 10^{-7}$
$Re \sim 10^8$
$Ek \sim 10^{-15}$

$Ro \sim 10^{-2}$
$Re \sim 10^{16}$
$Ek \sim 10^{-18}$

Image: E. Garnero

Axial vorticity, $Ek \sim 10^{-6}$ (Kageyama et al. Nature 2008)
Convective Rossby waves, still viscously controlled
Earth's Core: $l \sim DE^{1/3} \approx 10m$

Ocean dynamics

planetary - gyre scale $\sim O(1000) \text{ km}$

mesoscale $\sim O(100) \text{ km}$

submesoscale $\leq O(1) \text{ km}$

Langmuir Turbulence

open-ocean deep convection: mesoscale

- preconditioning
  - cyclonic gyre domes isopycnals, $L \sim 100 \text{ km}$
- deep convection
  - cooling events trigger deep plumes, $L \lesssim 1 \text{ km}, H \sim 2 \text{ km}, U \lesssim 10 \text{ cm/s}$
- lateral exchange
  - geostrophic eddies, $L \sim 10 \text{ km}$
- influenced by rotation
  - natural Rossby number $Ro^* \sim 0.1 - 0.4$

\[
Ro^* = \frac{L_{rot}}{H} = \left(\frac{B}{f^3 H^2}\right)^{1/2}
\]
Resolution of Ocean Component of Coupled IPCC models

Top-down approach: DNS not possible for several centuries!
Resolution of Ocean Component of Coupled IPCC models

Ocean Model Resolution (km)

Year

Approach to geostrophic turbulence bottom-up? .... middle-out?
Rotationally constrained (geostrophic) convective flows are highly anisotropic

- When do geometry and boundary conditions directly influence the small scales, and how might this be parameterized
  - theory ⇒ mechanical bc’s of secondary importance; Lab/DNS ⇒ hard to achieve high Re - Low Ro regimes

- When do boundary conditions influence the small scales via the dynamics of the large scales
  - vortex stretching in spherical geometries?

- Differences for LES closures to address in cartesian and spherical geometries
  - presently N/A

- When are boundary conditions unimportant?
  - thermal bl’s have surprising affect on rotational constraint

- Links between small scale and large scales
  - geostrophic convective turbulence very efficient at driving large-scale flows, sustained in a +ve feedback loop

- Where is the KE injected? Should the buoyancy force do significant work on the SGS?
  - to-date KE spectrum containing convection must be resolved, Low Ro challenge ⇒ separation of scales issue

- Magnetic Field/dynamos!

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Investigate simplified model scenarios
Rotating Rayleigh-Bénard Convection

\[ T = T_1 \]

Liquid: \( \nu, \kappa, \alpha \)

\[ T = T_0 \]

\[ \Omega \]

Convection

\[ R_a \propto \Delta T \text{ (thermal forcing)} \]

Conduction (No Fluid Motion)

\[ R_a \propto \Omega^2 \text{ (rotation rate)} \]

\[ R_{\text{crit}} \approx 8.7 T_0^{2/3} \text{ (linear stability)} \]

(Chandrasekhar, 1961)
Rotating Rayleigh-Bénard Convection

Laboratory Experiments are limited by engineering and fluid properties

Experiments:
- Rossby, JFM 1969
- Zhong, Ecke & Steinberg, JFM 1993
- Sakai, JFM 1997
- Vorobieff & Ecke, JFM 2002
- King, Stellmach, Noir, Hansen, & Aurnou, Nature 2009
- Kunnen, Guerts & Clerx, JFM 2010
- Zhong & Ahlers, JFM 2010
- Lui & Ecke, PRE 2011

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Sakai, JFM 1997: \( \text{Ra}^{1/3} = 36, \text{Ro} \approx 0.1, \sigma = 7 \)

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Widely held belief that rotationally constrained motions are strictly columnar
RRBC Results

Parameterization: dependence of global fluid properties on $[\text{Re(Ra)}, \text{Ro}, \text{Ek}, \text{Pr}]$
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- **Heat Transport - Nusselt Number**
  \[ Nu - 1 \propto \sigma^\alpha \left( \frac{Ra}{Ra_c} \right)^\beta \]

- **Flow Morphology**
  - Pathway to geostrophic turbulence at low Ro

\[ Ro_C = \sqrt{\frac{Ra}{\sigma} Ek} \]
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  - Low Ro transition to non-rotating scaling law
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UCLA group: courtesy Aurnou & Cheng

King et al Nature 2009
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- **Mixing**
  - Saturation of mean temp. gradient

  $$\partial_z T_{\text{mid}} \propto \sigma^\gamma(Ra_T/Ra_c)^\delta$$
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RRBC Results - Large Scale Flow Generation

Ra=10^9, Ro=0.36, σ = 7, Kunnen. et al JFM 2011

Convection appears to drive large scale (barotropic) dynamics.
Low Rossby Number Computational Challenge

- Fast waves + geostrophically balanced eddies limit DNS/Lab investigations

\[
\partial_t u + Ro^{-1} \hat{z} \times u \approx -Eu \nabla p, \quad \nabla \cdot u = 0
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\[ \partial_t u + Ro^{-1} \hat{z} \times u \approx -Eu \nabla p, \quad \nabla \cdot u = 0 \]

- Existence of reduced PDE models that filter fast waves and automatically enforce geostrophic balance?
Multi-Scale Asymptotics to the Rescue

- Exploit small parameters asymptotically (Ro, Ek): \( u = u_0 + Ro u_1 + \cdots \)

- Leading order balance, geostrophic approx’n (fast inertial waves filtered; Embid & Majda GAFD ’98)

\[
\begin{align*}
Ro^{-1} (\hat{z} \times u + \nabla p) & \approx 0 \\
\nabla \cdot u & = 0
\end{align*}
\]

\( \Rightarrow \) \( \nabla_\perp \cdot u_\perp \approx 0 \quad \partial_z (u_\perp, w, p) \approx 0 \)

T-P constraint

- Diagnostic solution: \( u \approx -\nabla \times \psi \hat{z} + w \hat{z}, \quad p = \psi \quad \zeta = \nabla_\perp^2 \psi \)

- Quasigeostrophic perturbation theory, solvability: \( \hat{z} \cdot \nabla \times , \quad \hat{z} \cdot \leftrightarrow f = \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} f dz \)

- Reinterpret Taylor-Proudman theory (MSA; KJ, Knobloch, Milliff & Werne, JFM 2006)
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\[ \partial_t \zeta + \mathbf{u}_\perp \cdot \nabla_\perp \zeta - \partial_Z w = \nabla_\perp^2 \zeta + \hat{\mathbf{z}} \cdot \nabla \times \bar{\mathbf{S}} \]

\[ \partial_Z \rightarrow \partial_Z + Ro \partial_Z \]

\[ \frac{D}{H} \sim Ro \]
Quasigeostrophic Rayleigh-Bénard Convection

- Four Flow Regimes as $Ra$ ↑
  - CTC’s give way to GT (columnar flow not the end state!)

- Turbulent Inverse Cascade (Julien et al. GAFD 2012)
  - GT drives large scale barotropic vortices (jets on f-plane?)

- Turbulent Heat Transport Scaling Law (Julien et al. PRL 2012)
  - GT interior restricts turbulent HT NOT thermal BL’s
Quasigeostrophic Rayleigh-Bénard Convection

\[ g = -\hat{Z} \]

\[ \partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla^2 \perp \zeta \]

\[ \partial_t w + J[\psi, w] + \partial_Z \psi = \nabla^2 \perp w + \frac{Ra}{\sigma} \overline{\theta} \]

\[ \partial_t \overline{\theta} + J[\psi, \overline{\theta}] + w \partial_Z \langle T \rangle = \frac{1}{\sigma} \nabla^2 \perp \overline{\theta} \]

\[ \partial_Z \langle w \overline{\theta} \rangle = \frac{1}{\sigma} \partial_Z Z \langle T \rangle \]

J. et al JFM 2006, GAFD ‘12

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\[ \text{RaE}^{4/3} = 100, \ Pr = 1 \]

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Heimpel et al, Nature ’05

Calkins, Julien, Rubio ’13
Quasigeostrophic Rayleigh-Bénard Convection

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\[
Nu - 1 = C_1 \sigma^{-1/2} Ra^{3/2} E^2
\]
Low Ro Heat Transfer:

\[ Nu - 1 = \frac{1}{25} \sigma^{-\frac{1}{2}} \left( RaE^{\frac{4}{3}} \right)^{\frac{3}{2}} \]

Nondimensional \#'s:

\[ Nu \equiv \frac{QH}{\rho_0 c_p \kappa \Delta T}, \quad Ra = \frac{g \alpha \Delta TH^3}{\nu \kappa}, \quad E = \frac{\nu}{fH^2} \]

\[ \sigma = \frac{\nu}{\kappa} \]

Convective Taylor Columns

Ultimate Heat Transport Scaling Law
Ultimate Heat Transport Scaling Law

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Low Ro Heat Transfer: \[ Nu - 1 = \frac{1}{25} \sigma^{-\frac{1}{2}} \left( RaE^{\frac{4}{3}} \right)^{\frac{3}{2}} \]

- turbulent interior controls heat transport (GL theory)

\[ \mathcal{E}_\theta \approx \mathcal{E}_\theta^{int} = \left\langle |\partial_z T|^2 \right\rangle + \left\langle |\nabla \perp \theta|^2 \right\rangle \equiv Nu \]

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3D Quasigeostrophic-\(\beta\) convection

\[ g = \hat{r} \]

- 3DQG-\(\beta\) convection valid for \(O(1)\) slopes
  - strong vertical motions, \(w \sim O(u)\)
- Linear Stability: Fundamental mode is the Busse mode \(\text{(Busse, JFM '70)}\)
  - Vertically invariant Busse regime recaptured as \(\chi \to 0\), modulation otherwise
- New 3D Rossby modes of propagation
  - Dynamics are fundamentally three dimensional
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\partial_t w + J[\psi, w] + \left( \frac{\beta}{\tan \chi} \right)^2 \partial_Z \psi = \nabla^2 \perp w
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\partial_t T + J[\psi, T] = \frac{1}{\sigma} \nabla^2 \perp T
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BC : \[ \tilde{w} \mp \left( \frac{\tan \chi}{AH E} \right) \partial_y \psi = 0 \]

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Calkins, J, Marti, JFM '13
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\(\chi = 45^\circ\)

Mode 1

Mode 2

Mode 3

Calkins, J, Marti, JFM '13
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  Dynamics are fundamentally three dimensional! Dynamics cannot be treated two dimensionally

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\[ g = \hat{r} \]

\[ \partial_t \zeta + J[\psi, \zeta] - \partial_Z w = \nabla_\perp^2 \zeta + \frac{Ra}{\sigma 16} \partial_y T \]

\[ \partial_t w + J[\psi, w] + \left( \frac{\beta}{\tan \chi} \right)^2 \partial_Z \psi = \nabla_\perp^2 w \]

\[ \partial_t T + J[\psi, T] = \frac{1}{\sigma} \nabla_\perp^2 T \]

BC: \[ \tilde{w} \mp \left( \frac{\tan \chi}{A H E} \right) \partial_y \psi = 0 \]

- 3DQG-$\beta$ convection valid for $O(1)$ slopes
  - Strong vertical motions, $w \sim O(u)$

- Linear Stability: Fundamental mode is the Busse mode (Busse, JFM '70)
  - Vertically invariant Busse regime recaptured as $\chi \to 0$, modulation otherwise

- New 3D Rossby modes of propagation
  - Dynamics are fundamentally three dimensional! Dynamics cannot be treated two dimensionally

Calkins, J, Marti, JFM '13
\[ \bar{U} = \hat{Z} \times \nabla_\perp \bar{P}, \quad \bar{\Theta} = \partial_Z \bar{P} \]

\[
\left( \frac{\partial}{\partial t} + \bar{U} \cdot \nabla \right) \partial_Z \bar{P} = 0
\]

\[
\frac{\partial \langle \bar{U} \rangle}{\partial t} + \nabla \cdot \langle \bar{U} \otimes \bar{U} + u \otimes u \rangle = -\nabla \langle \bar{\Pi} \rangle
\]

Baroclinic Dynamics \hspace{2cm} Barotropic Dynamics

Non-Hydrostatic Dynamics

\[ \hat{z} \times u_\perp = -\nabla_\perp p, \quad p = -\psi \]

\[
(\partial_t + \bar{U} \cdot \nabla_\perp) \nabla_\perp^2 \psi + J(\psi, \nabla_\perp^2 \psi) + \partial_Z w = \frac{1}{\text{Re}} \nabla_\perp^4 \psi
\]

\[
(\partial_t + \bar{U} \cdot \nabla_\perp) w + J(\psi, w) - \partial_Z \psi = \theta + \frac{1}{\text{Re}} \nabla_\perp^2 w
\]

\[
(\partial_t + \bar{U} \cdot \nabla_\perp) \theta + J(\psi, \theta) + \nabla_\perp \psi \cdot \partial_Z \bar{U} + w \partial_Z \bar{\Theta} = \frac{1}{\text{Pe}} \nabla_\perp^2 \theta
\]

\[
\left( \frac{\partial}{\partial \tau} - \frac{1}{\text{Pe}} \partial_Z^2 \right) \bar{T} = -\partial_Z \bar{F}
\]

Global Mean Temperature & Flux

\[
\bar{T} = \lim_{\bar{t} \to \infty} \frac{1}{\bar{t}} \int_0^\bar{t} \frac{1}{|A|} \int_A \bar{\Theta} \, dX \, dY \, d\bar{t}'
\]

\[
\bar{F} = \lim_{\bar{t} \to \infty} \frac{1}{\bar{t}} \int_0^\bar{t} \frac{1}{|A|} \left[ \int_A \bar{w} \theta \, dX \, dY - \int_{\partial A} \bar{U} \cdot \bar{d}l \right] \, d\bar{t}'
\]
Outlook for 3D QG

- Reduced PDE's well suited to QG dynamics, computationally less challenging.
- Incompressible aDNS ("a"symptotic)
  - Investigate route to turbulence: columnar breakdown
  - Mean flow generation: inverse turbulent cascade?
  - Efficiency of heat transport: scaling laws
- Anelastic (stratification) aDNS Simulations
- Coupling to reduced planetary scale dynamics, required by MHD

Sprague et al JFM '06  Groom et al PRL '10  Julien et al GAFD '12  Julien et al PRL '12